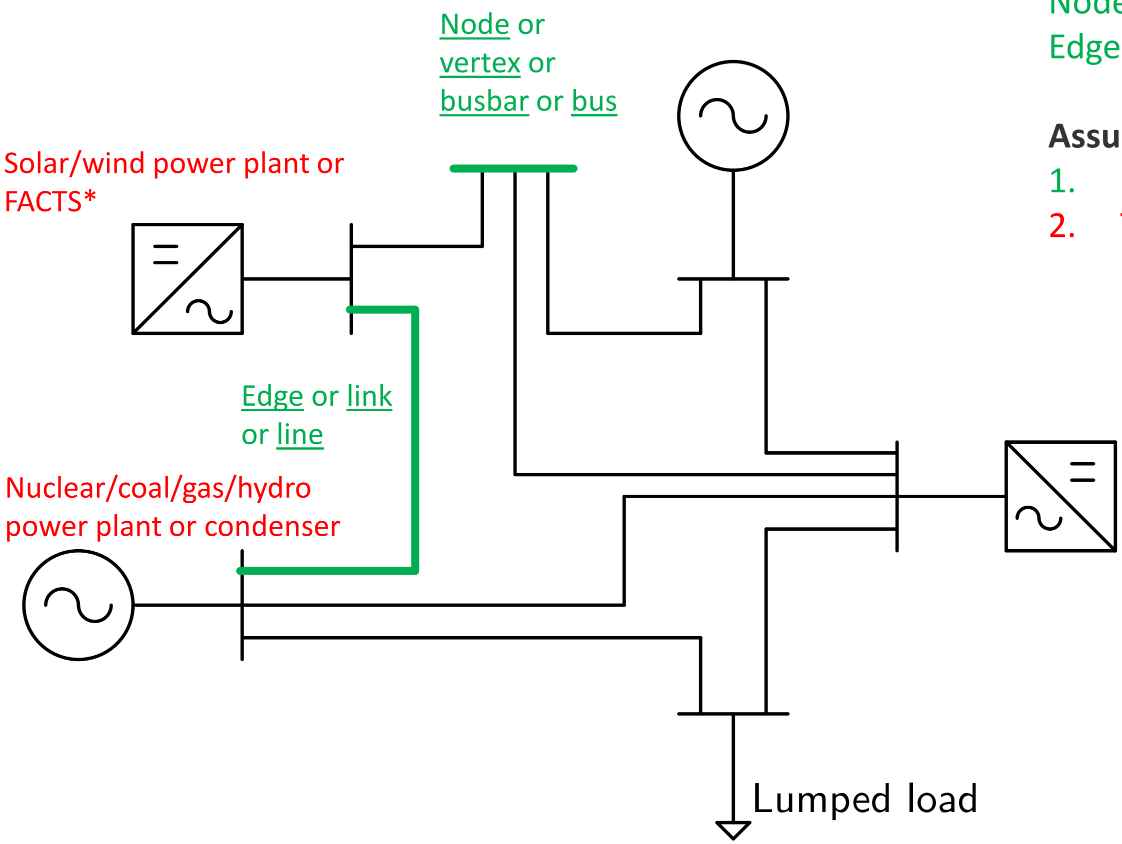
Data-centric approach to capture non-polynomial nonlinear dynamics

Marcos Netto
Research Engineer and Director’s Postdoctoral Fellow

Collaborators: Yoshihiko Susuki (Kyoto University), Venkat Krishnan (PA Consulting), Yingchen ‘YC’ Zhang (Utilidata)

NREL Autonomous Energy Systems Workshop, July 13–15, 2022

Background



Node → Substation
Edge → Transmission line

Assumptions:

1. Nodes and edges are passive elements
2. The dynamics are mainly driven by generators responding to events

Fact:

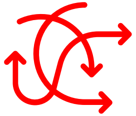
- Complex nonlinear dynamics might emerge from the interaction of multiple elements coupled through a meshed network

*FACTS: Flexible Alternating Current Transmission System

Motivation

- First-principles models available for offline planning and design

- These models are reliable but not suitable for real-time control — *nonlinear models with thousands of state variables*



- Skepticism on the application of black-box approaches to critical infrastructures

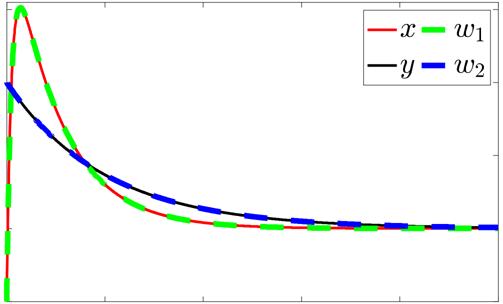
(We learned later that) the above holds for other engineering systems than power systems, e.g., robotic systems

Koopman operator formalism is attractive because

- It offers a linear representation of the underlying nonlinear dynamics — *suitable for real-time control*
- This linear representation preserves physical interpretability

Motivation

Lifted system	✗ Infinite dimensional	✓ Linear
Original system	✓ Finite dimensional	✗ Nonlinear



Consider the *nonlinear* dynamical system

$$\frac{dx}{dt} = \lambda_1 \cdot (x - y^2),$$
$$\frac{dy}{dt} = \lambda_2 \cdot y, \qquad (1)$$

$\boldsymbol{x} = [x\ y]^\top \in \mathbb{R}^2$ is the **state**, λ_1, λ_2 are scalars.

Lifting

Define **observables**

$$g_1(\boldsymbol{x}) = w_1 = x,$$
$$g_2(\boldsymbol{x}) = w_2 = y,$$
$$g_3(\boldsymbol{x}) = w_3 = y^2,$$
$$\boldsymbol{g}(\boldsymbol{x}) = [g_1\ g_2\ g_3]^\top \in \mathbb{R}^3.$$

$\rightarrow$

$$\frac{dw_1}{dt} = \frac{dx}{dt} = \lambda_1 \cdot (x - y^2) = \lambda_1 \cdot (w_1 - w_3),$$
$$\frac{dw_2}{dt} = \frac{dy}{dt} = \lambda_2 \cdot y = \lambda_2 \cdot w_2,$$
$$\frac{dw_3}{dt} = 2y \cdot \frac{dy}{dt} = 2y \cdot (\lambda_2 \cdot y) = 2\lambda_2 \cdot y^2 = 2\lambda_2 \cdot w_3,$$

and we have the *linear* dynamical system

$$\begin{bmatrix} \frac{dw_1}{dt} \\ \frac{dw_2}{dt} \\ \frac{dw_3}{dt} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & -\lambda_1 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 2\lambda_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \qquad (2)$$

Discrete time dynamical system

Consider the discrete time dynamical system

$$\boldsymbol{x}_{k+1} = \boldsymbol{T}(\boldsymbol{x}_k) \tag{3}$$

where:

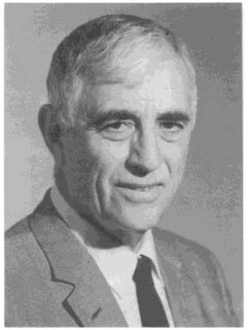
- $\boldsymbol{x}$ is the state, an element of the state space $S \subset \mathbb{R}^n$
- $\boldsymbol{T} : S \mapsto S$ is a map
- $k \in \mathbb{Z}^+ \cup \{0\}$ is the time index

Define:

$$g : S \mapsto \mathbb{R}^1 \tag{4}$$

an **observable** on this dynamical system. The space of observables is infinite

Koopman operator



The **Koopman operator**, U, is a linear transformation on the vector space of observables

$$Ug(\boldsymbol{x}) = g \circ \boldsymbol{T}(\boldsymbol{x}) \tag{5}$$

The **KO is infinite-dimensional** because the space of observables is infinite

The **KO is linear** because of the linearity of the composition operation

$$U(g_1 + g_2)(\boldsymbol{x}) = (g_1 + g_2) \circ \boldsymbol{T}(\boldsymbol{x}) = g_1 \circ \boldsymbol{T}(\boldsymbol{x}) + g_2 \circ \boldsymbol{T}(\boldsymbol{x}) = Ug_1(\boldsymbol{x}) + Ug_2(\boldsymbol{x})$$

The **KO exists** as long as $\boldsymbol{T}$ exists, **and it is unique** as long as $\boldsymbol{T}$ is unique

(Koopman, 1932) for original notion of KO for Hamiltonian systems

Numerical estimation of the Koopman operator – EDMD method

Define:

- **Data matrices:**

$$\boldsymbol{X} = [\boldsymbol{x}_1 \ \ldots \ \boldsymbol{x}_N] \quad \text{and} \quad \boldsymbol{X}^+ = [\boldsymbol{x}_2 \ \ldots \ \boldsymbol{x}_{N+1}] \tag{6}$$

- **Vector of observable functions:**

$$\boldsymbol{g}\left(\boldsymbol{x}_k\right) = \left[g_1\left(\boldsymbol{x}_k\right); \ \ldots; \ g_q\left(\boldsymbol{x}_k\right)\right]^\top \tag{7}$$

$\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^q$, $q > n$.

- **Observable matrices:**

$$\boldsymbol{O}_{\boldsymbol{X}} = \left[\boldsymbol{g}\left(\boldsymbol{x}_1\right) \ \ldots \ \boldsymbol{g}\left(\boldsymbol{x}_N\right)\right], \quad \boldsymbol{O}_{\boldsymbol{X}^+} = \left[\boldsymbol{g}\left(\boldsymbol{x}_2\right) \ \ldots \ \boldsymbol{g}\left(\boldsymbol{x}_{N+1}\right)\right] \tag{8}$$

A finite-dimensional approximation to the Koopman operator is estimated as

$$\boldsymbol{K} = \boldsymbol{O}_{\boldsymbol{X}^+}\boldsymbol{O}_{\boldsymbol{X}}^\dagger, \qquad \boldsymbol{K} \in \mathbb{R}^{q \times q} \tag{9}$$

A few observations

The former numerical procedure

- performs well when the underlying nonlinear dynamics are polynomial
- has challenges identifying more general dynamics, especially when it involves the composition of nonlinear functions

Questions

- Can we take advantage of first-principles models to guide the selection of observable functions?

- If affirmative, is there any advantage in this physics-aware selection of observable functions?

Key idea

- Find an intermediate transformation to eliminate non-polynomial terms without approximations

Main result

Consider a dynamical system of this form:

$$\dot{x}_i = \boldsymbol{k}_0^\top \boldsymbol{x} + k_1 h_1(\boldsymbol{x}) + \cdots + k_m h_m(\boldsymbol{x})$$

Main result

Consider a dynamical system of this form:

$$\dot{x}_i = \boldsymbol{k}_0^\top \boldsymbol{x} + k_1 h_1(\boldsymbol{x}) + \cdots + k_m h_m(\boldsymbol{x})$$

Elementary functions

$$h(x) = e^x$$
$$h(x) = \frac{1}{b+x}$$
$$h(x) = x^b$$
$$h(x) = \ln(x)$$
$$h(x) = \sin(x)$$

Composition of elementary functions

$$h(x) = \frac{1}{1+e^{-x}}$$ ← Sigmoid function

$$h(\boldsymbol{x}) = x_1 \cdot \cos(x_2)$$
$$h(\boldsymbol{x}) = \sqrt{x_1^2 + x_2^2}$$

Main result

Consider a dynamical system of this form:

$$\dot{x}_i = \boldsymbol{k}_0^\mathsf{T}\boldsymbol{x} + k_1 h_1(\boldsymbol{x}) + \cdots + k_m h_m(\boldsymbol{x})$$

The lifting procedure is as follows:

$$\dot{x}_i = \boldsymbol{k}_0^\mathsf{T}\boldsymbol{x} + k_1 z_1 + \cdots + k_m z_m,$$
$$\dot{z}_i = \mathcal{L}_{\boldsymbol{f}} h_i(\boldsymbol{x}),$$

$$\mathcal{L}_{\boldsymbol{f}} h_i(\boldsymbol{x}) = \frac{\partial h_i(\boldsymbol{x})}{\partial x_1}\dot{x}_1 + \cdots + \frac{\partial h_i(\boldsymbol{x})}{\partial x_n}\dot{x}_n.$$

TABLE I
TRANSFORMATIONS FOR UNIVARIATE ELEMENTARY FUNCTIONS

Elementary function	New variable(s)	New differential equation(s)
$h(x) = e^x$	$z = e^x$	$\dot{z} = e^x\dot{x} = z\dot{x}$
$h(x) = \frac{1}{b+x}$	$z = \frac{1}{b+x}$	$\dot{z} = -\frac{1}{(b+x)^2}\dot{x} = -z^2\dot{x}$
$h(x) = \ln x$	$z_1 = \ln x$ $z_2 = x^{-1}$	$\dot{z} = x^{-1}\dot{x} = z_2\dot{x}$ $\dot{z}_2 = -x^{-2}\dot{x} = -z_2^2\dot{x}$
$h(x) = \sin x$	$z_1 = \sin x$ $z_2 = \cos x$	$\dot{z}_1 = (\cos x)\dot{x} = z_2\dot{x}$ $\dot{z}_2 = (-\sin x)\dot{x} = -z_1\dot{x}$

TABLE II
EXAMPLES OF POLYNOMIALIZATION OF SYSTEMS GIVEN BY COMPOSITION OF ELEMENTARY FUNCTIONS

Original system	New variables	Lifted system
$\dot{x} = \frac{1}{1+e^{-x}}$		$\dot{x} = z_2$
	$z_1 = e^{-x}$	$\dot{z}_1 = -e^{-x}\frac{1}{1+e^{-x}} = -z_1 z_2$
	$z_2 = \frac{1}{1+z_1}$	$\dot{z}_2 = -\frac{1}{(1+z_1)^2}(-z_1 z_2) = z_1 z_2^3$
$\dot{x} = x\cos x$		$\dot{x} = z_2$
	$z_1 = \cos x$	$\dot{z}_1 = -\sin x(x\cos x) = -z_2 z_3 = -z_1 z_4$
	$z_2 = x z_1$	$\dot{z}_2 = x\cos x\cos x - x^2\sin x\cos x$ $= z_1 z_2 - z_2 z_4$
	$z_3 = \sin x$	$\dot{z}_3 = \cos x(x\cos x) = z_1 z_2$
	$z_4 = x z_3$	$\dot{z}_4 = x\cos x\sin x + x^2\cos x\cos x$ $= z_2 z_3 + z_2^2 = z_1 z_4 + z_2^2$

Main result

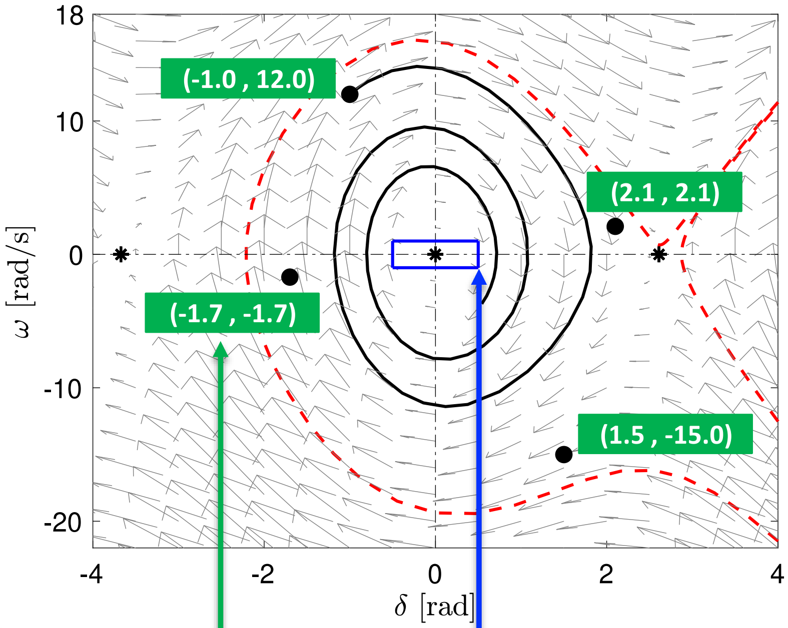
Consider a dynamical system of this form:

$$\dot{x}_i = \boldsymbol{k}_0^{\mathsf{T}} \boldsymbol{x} + k_1 h_1(\boldsymbol{x}) + \cdots + k_m h_m(\boldsymbol{x})$$

Example with simplified power system model:

$$\dot{\delta} = \omega,$$

$$\dot{\omega} = \frac{1}{M}\left(k_1 + k_2 \cos\delta + k_3 \sin\delta - \frac{D}{\omega_s}\omega\right)$$



'Learning' on a linear region of the state space

Testing on a nonlinear region of the state space, close to the stability boundary

$$z_1 := \delta$$
$$z_2 := \omega$$
$$z_3 := \sin\delta$$
$$z_4 := \cos\delta$$

$$\dot{z}_1 = z_2,$$

$$\dot{z}_2 = \frac{1}{M}\left(k_1 + k_2 z_4 + k_3 z_3 - \frac{D}{\omega_s} z_2\right)$$

$$\dot{z}_3 = \mathcal{L}_{\boldsymbol{f}} \sin\delta = \frac{\partial \sin\delta}{\partial \delta}\dot{\delta} + \frac{\partial \sin\delta}{\partial \omega}\dot{\omega} = z_2 z_4,$$

$$\dot{z}_4 = \mathcal{L}_{\boldsymbol{f}} \cos\delta = \frac{\partial \cos\delta}{\partial \delta}\dot{\delta} + \frac{\partial \cos\delta}{\partial \omega}\dot{\omega} = -z_2 z_3,$$

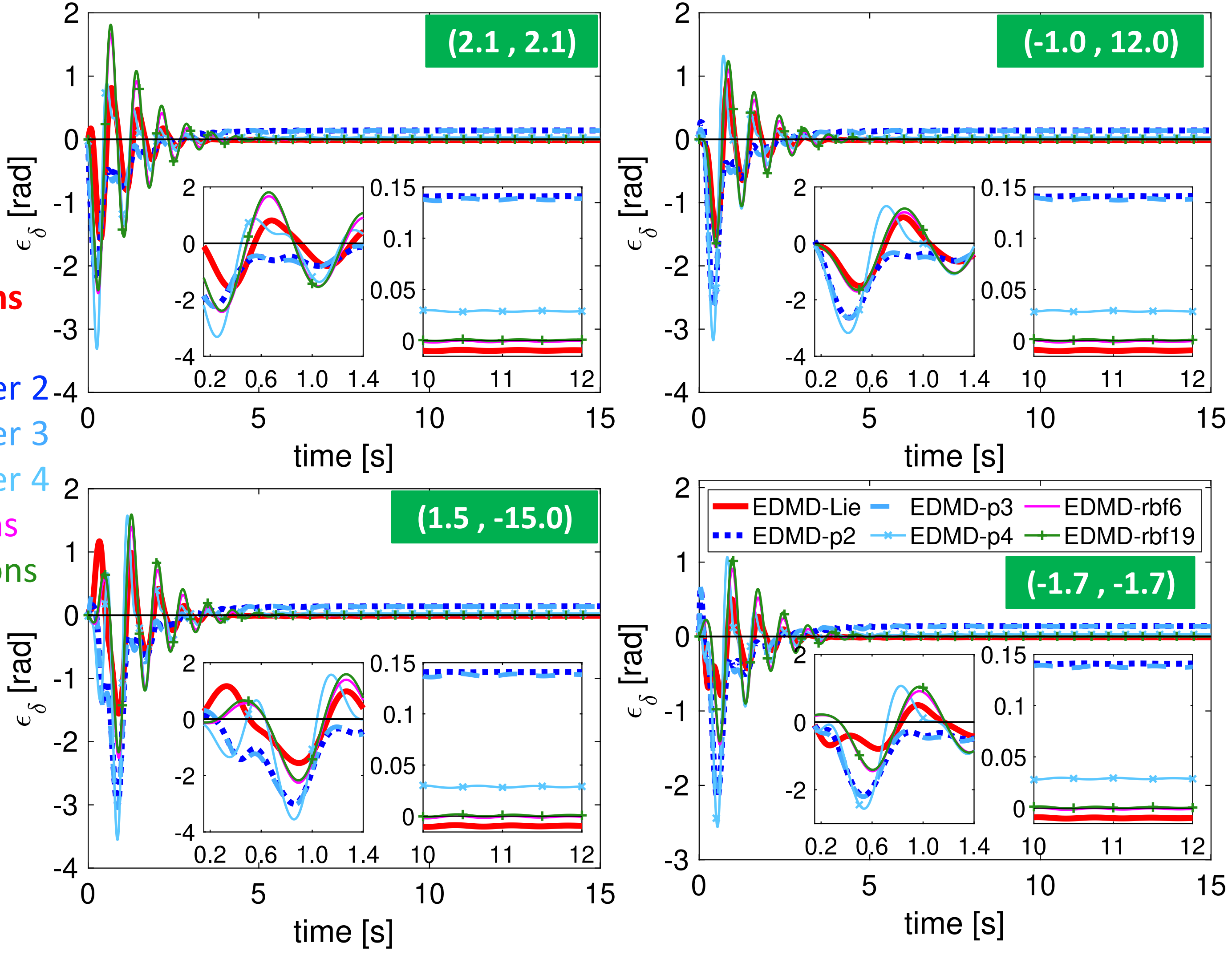
$$\{\delta\,,\ \omega\,,\ \sin\delta\,,\ \cos\delta\,,\ \omega\cos\delta\,,\ -\omega\sin\delta\,\}$$

Numerical results

Lie – proposed method
6 observable functions

p2 – monomials up to order 2
p3 – monomials up to order 3
p4 – monomials up to order 4
rbf6 – 4 radial basis functions
rbf19 – 17 radial basis functions

- Lie is more accurate than p2, p3, and p4 in all cases.
- Lie achieves same level of accuracy of rbf6 and rbf19 with less or equal number of observable functions.



Example of a robotic system

$$
\begin{aligned}
\dot{x} &= v_x \cos(\psi) - v_y \sin(\psi), \\
\dot{z} &= v_x \sin(\psi) + v_y \cos(\psi), \\
\dot{\psi} &= \omega, \\
\dot{v}_x &= k_1 v_y \omega + k_2 v_x \sqrt{v_x^2 + v_y^2} + k_3 v_y \sqrt{v_x^2 + v_y^2} \arctan\left(\frac{v_y}{v_x}\right) + k_4 u_1, \\
\dot{v}_y &= k_5 v_x \omega + k_6 v_y \sqrt{v_x^2 + v_y^2} + k_7 v_x \sqrt{v_x^2 + v_y^2} \arctan\left(\frac{v_y}{v_x}\right) + k_8 u_2, \\
\dot{\omega} &= k_9 v_x v_y + \boxed{k_{10} \mathrm{sgn}(\omega) \omega^2} + k_{11} u_2,
\end{aligned}
$$

$$
\begin{aligned}
\dot{x} &= v_x \cos(\psi) - v_y \sin(\psi), \\
\dot{z} &= v_x \sin(\psi) + v_y \cos(\psi), \\
\dot{\psi} &= \omega, \\
\dot{v}_x &= k_1 v_y \omega + k_2 v_x \sqrt{v_x^2 + v_y^2} + k_3 v_y \sqrt{v_x^2 + v_y^2} \frac{\cos(v_y/v_x)}{\sin(v_y/v_x)} + k_4 u_1, \\
\dot{v}_y &= k_5 v_x \omega + k_6 v_y \sqrt{v_x^2 + v_y^2} + k_7 v_x \sqrt{v_x^2 + v_y^2} \frac{\cos(v_y/v_x)}{\sin(v_y/v_x)} + k_8 u_2, \\
\dot{\omega} &= k_9 v_x v_y + \boxed{k_{10} \omega^2 \frac{e^{2\omega} - 1}{e^{2\omega} + 1}} + k_{11} u_2,
\end{aligned}
$$

[1] G. Mamakoukas, M. Castano, X. Tan, and T. D. Murphey, “Local Koopman operators for data-driven control of robotic systems,” in Robotics: Science and Systems, 2019, p. 54.
[2] Supplementary material for *M. Netto, Y. Susuki, V. Krishnan and Y. Zhang, "On Analytical Construction of Observable Functions in Extended Dynamic Mode Decomposition for Nonlinear Estimation and Prediction," in IEEE Control Systems Letters, vol. 5, no. 6, pp. 1868-1873, Dec. 2021.*

Example of a robotic system – Summary

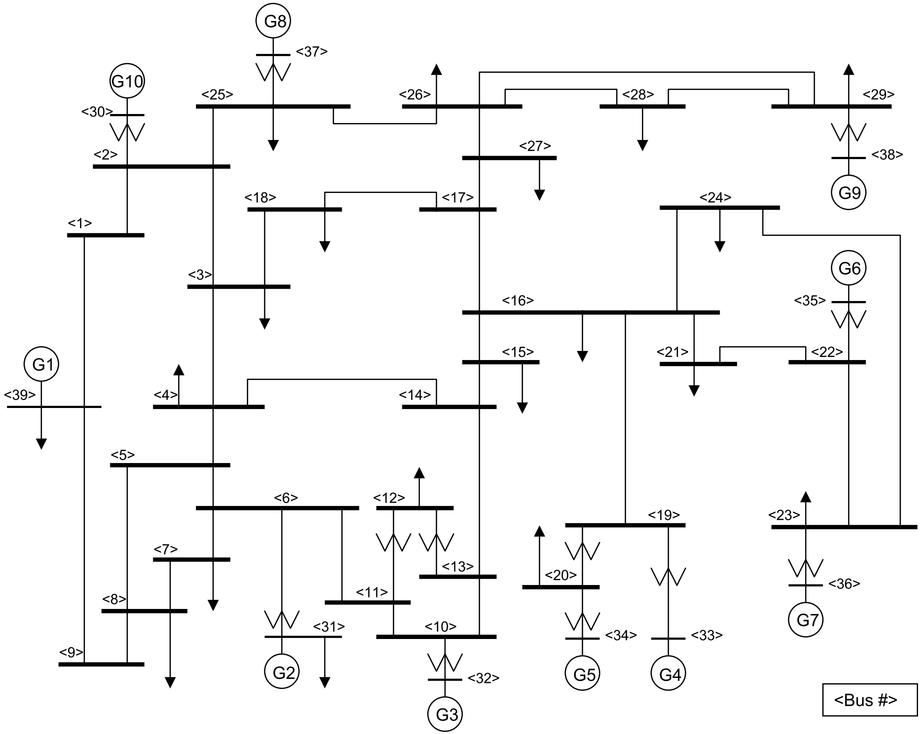
- *Original system* is of dimension 6
- *Lifted system* is of dimension 23
- At the end, we obtained 102 observable functions

Any dynamical system of this form

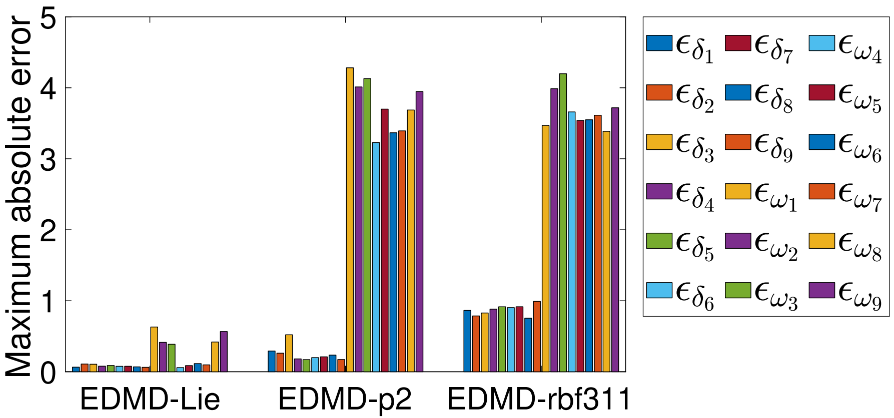
$$\dot{x}_i = \boldsymbol{k}_0^\mathsf{T}\boldsymbol{x} + k_1 h_1(\boldsymbol{x}) + \cdots + k_m h_m(\boldsymbol{x})$$

can be lifted into a quadratic polynomial system

Example of a multimachine power system



$$
\left.
\begin{aligned}
&\frac{\mathrm{d}\delta_i}{\mathrm{d}t} = \omega_i, \\
&\frac{H_i}{\pi f_\mathrm{b}}\frac{\mathrm{d}\omega_i}{\mathrm{d}t} = -D_i\omega_i + P_{\mathrm{m}i} - G_{ii}E_i^2 \\
&- \sum_{j=1, j\neq i}^{10} E_iE_j\left\{G_{ij}\cos(\delta_i-\delta_j) + B_{ij}\sin(\delta_i-\delta_j)\right\}
\end{aligned}
\right\}
$$



Ongoing work based on industry models

$$T'_{doi}\frac{dE'_{qi}}{dt} = -\left(X_{di} - X'_{di}\right)\left(\sum_{k=1}^{m} G'_{\underset{ik}{\text{red}}}\left[E'_{dk}\cos\left(\delta_k - \delta_i\right) - E'_{qk}\sin\left(\delta_k - \delta_i\right)\right] - B'_{\underset{ik}{\text{red}}}\left[E'_{dk}\sin\left(\delta_k - \delta_i\right) + E'_{qk}\cos\left(\delta_k - \delta_i\right)\right]\right) - E'_{qi} + E_{fdi} \quad i = 1, \ldots, m \tag{1}$$

$$T'_{qoi}\frac{dE'_{di}}{dt} = \left(X_{qi} - X'_{qi}\right)\left(\sum_{k=1}^{m} G'_{\underset{ik}{\text{red}}}\left[E'_{dk}\sin\left(\delta_k - \delta_i\right) + E'_{qk}\cos\left(\delta_k - \delta_i\right)\right] + B'_{\underset{ik}{\text{red}}}\left[E'_{dk}\cos\left(\delta_k - \delta_i\right) - E'_{qk}\sin\left(\delta_k - \delta_i\right)\right]\right) - E'_{di} \quad i = 1, \ldots, m \tag{2}$$

$$\frac{d\delta_i}{dt} = \omega_i - \omega_s \quad i = 1, \ldots, m \tag{3}$$

$$\frac{2H_i}{\omega_s}\frac{d\omega_i}{dt} = -\left(\sum_{k=1}^{m} G'_{\underset{ik}{\text{red}}}\left[E'_{di}E'_{dk}\cos\left(\delta_k - \delta_i\right) - E'_{di}E'_{qk}\sin\left(\delta_k - \delta_i\right)\right] - B'_{\underset{ik}{\text{red}}}\left[E'_{di}E'_{dk}\sin\left(\delta_k - \delta_i\right) + E'_{di}E'_{qk}\cos\left(\delta_k - \delta_i\right)\right]\right) - \left(\sum_{k=1}^{m} G'_{\underset{ik}{\text{red}}}\left[E'_{qi}E'_{dk}\sin\left(\delta_k - \delta_i\right) + E'_{qi}E'_{qk}\cos\left(\delta_k - \delta_i\right)\right] + B'_{\underset{ik}{\text{red}}}\left[E'_{qi}E'_{dk}\cos\left(\delta_k - \delta_i\right) - E'_{qi}E'_{qk}\sin\left(\delta_k - \delta_i\right)\right]\right) + T_{Mi} - T_{FWi} \quad i = 1, \ldots, m \tag{4}$$

Ongoing work based on industry models

$$T_{Ei}\frac{dE_{fdi}}{dt} = -K_{Ei}E_{fdi} - A_{xi}E_{fdi}e^{B_{xi}E_{fdi}} + V_{Ri} \quad i=1,\ldots,m \tag{5}$$

$$T_{Fi}\frac{dR_{fi}}{dt} = -R_{fi} + \frac{K_{Fi}}{T_{Fi}}E_{fdi} \quad i=1,\ldots,m \tag{6}$$

$$\begin{aligned} T_{Ai}\frac{dV_{Ri}}{dt} = & -K_{Ai}\Bigg\{ \Bigg[E'_{di} - \Bigg(\sum_{k=1}^{m} R_{si}G'_{\substack{\text{red}\\ik}} \left[E'_{dk}\cos(\delta_k-\delta_i) - E'_{qk}\sin(\delta_k-\delta_i)\right] - R_{si}B'_{\substack{\text{red}\\ik}} \left[E'_{dk}\sin(\delta_k-\delta_i) + E'_{qk}\cos(\delta_k-\delta_i)\right] \Bigg) \\ & + \Bigg(\sum_{k=1}^{m} X'_{di}G'_{\substack{\text{red}\\ik}} \left[E'_{dk}\sin(\delta_k-\delta_i) + E'_{qk}\cos(\delta_k-\delta_i)\right] + X'_{di}B'_{\substack{\text{red}\\ik}} \left[E'_{dk}\cos(\delta_k-\delta_i) - E'_{qk}\sin(\delta_k-\delta_i)\right] \Bigg) \Bigg]^2 \\ & + \Bigg[E'_{qi} - \Bigg(\sum_{k=1}^{m} R_{si}G'_{\substack{\text{red}\\ik}} \left[E'_{dk}\sin(\delta_k-\delta_i) + E'_{qk}\cos(\delta_k-\delta_i)\right] + R_{si}B'_{\substack{\text{red}\\ik}} \left[E'_{dk}\cos(\delta_k-\delta_i) - E'_{qk}\sin(\delta_k-\delta_i)\right] \Bigg) \\ & - \Bigg(\sum_{k=1}^{m} X'_{di}G'_{\substack{\text{red}\\ik}} \left[E'_{dk}\cos(\delta_k-\delta_i) - E'_{qk}\sin(\delta_k-\delta_i)\right] - X'_{di}B'_{\substack{\text{red}\\ik}} \left[E'_{dk}\sin(\delta_k-\delta_i) + E'_{qk}\cos(\delta_k-\delta_i)\right] \Bigg) \Bigg]^2 \Bigg\}^{1/2} \\ & - V_{Ri} + K_{Ai}R_{fi} - \frac{K_{Ai}K_{Fi}}{T_{Fi}}E_{fdi} + K_{Ai}V_{\text{ref}i} \quad i=1,\ldots,m \end{aligned} \tag{7}$$

$$T_{CHi}\frac{dT_{Mi}}{dt} = -T_{Mi} + P_{SVi} \quad i=1,\ldots,m \tag{8}$$

$$T_{SVi}\frac{dP_{SVi}}{dt} = -P_{SVi} + P_{Ci} - \frac{1}{R_{Di}}\left(\frac{\omega_i}{\omega_s} - 1\right) \quad i=1,\ldots,m \tag{9}$$

Conclusions and ongoing work

- We **developed and demonstrated an analytical procedure to construct observable functions for extended dynamic mode decomposition.**
- Beyond Koopman operator theory, the lifting method based on Lie derivatives is a general mathematical tool and might have interesting applications in other domains.
  - **Example: Convexification of nonlinear, non-convex problems.**
- We are developing a general routine in MATLAB using the Symbolic Math Toolbox. This computational tool will **automate the analytical construction of observable functions.**
- Soon, we expect to have a numerical example with an **industry-based power system model that includes synchronous and doubly-fed induction generators.**

Thank you

Marcos Netto
marcos.netto@nrel.gov

Acknowledgment: Work funded by NREL Director’s Fellowship program under the Laboratory Directed Research and Development (LDRD).

NREL/JA-5D00-77706

www.nrel.gov

NREL/JA-5D00-77706

This work was authored in part by the National Renewable Energy Laboratory (NREL), operated by Alliance for Sustainable Energy, LLC, for the U.S. Department of Energy (DOE) under Contract No. DE-AC36-08GO28308. Funding provided by the Laboratory Directed Research and Development (LDRD) program at NREL. The views expressed in the article do not necessarily represent the views of the DOE or the U.S. Government. The U.S. Government retains and the publisher, by accepting the article for publication, acknowledges that the U.S. Government retains a nonexclusive, paid-up, irrevocable, worldwide license to publish or reproduce the published form of this work, or allow others to do so, for U.S. Government purposes.

NREL Transforming ENERGY