

The curled wake model: equivalence of shed vorticity models

Luis A. Martínez-Tossas and Emmanuel Branlard

National Renewable Energy Laboratory, Golden, Colorado, USA

E-mail: luis.martinez@nrel.gov

Abstract. In this work, we explore the spanwise velocities in the wake of a yawed wind turbine. In the curled wake model, these motions are induced by a collection of vortices shed from the rotor plane. The direction of the vorticity generated by yaw is aligned with the main flow (streamwise) direction. The streamwise vorticity induces velocities in the spanwise directions. These are the motions responsible for creating the curled wake mechanism. In this work, we explore a more accurate formulation for this mechanism, using a vortex cylinder. Under certain assumptions, the new and original curled wake models yield the same mathematical formulation. Also, both models predict an elliptic distribution of vortex strength, where the main difference is the location of the vortices.

1. Introduction

When a wind turbine is yawed with respect to the inflow velocity, a curled wake profile is produced. The curled wake mechanism has been explained in the literature as a collection of counter-rotating vortices shed from the rotor plane [1, 2, 3, 4]. This explanation is analogous to the formation of vortices along the span of a wing and has been shown to agree well with numerical simulations and experimental measurements [4]. Previous work on the wake of a yawed wind turbine used a rigid, semi-infinite vortex cylinder representation of the wake. This model was first introduced by Coleman et al. [5], and different improvements followed [6, 7, 8]. However, the generation of the vortices is not well understood, and there is room for new explanations to better understand the curled wake mechanism and improve current models. In this work, we use a vorticity formulation to understand how when the wind turbine is yawed, a streamwise component of vorticity is generated. This component of vorticity induces velocities in a plane parallel to the rotor, deforming the wake and creating the curled wake shape.

The original explanation of the vortices created by a wind turbine in yaw states that a collection of vortices is distributed along a straight line, going from the bottom to the top of the rotor [1, 2, 3, 4]. These vortices are aligned with streamwise direction and are convected downstream. Figure 1 (left) shows a schematic of this mechanism. If we think of the wind turbine wake as an inclined vortex cylinder, the projection of the tangential vorticity component onto the streamwise direction yields a collection of vortices along the circumference of the rotor. Figure 1 (right) shows a schematic of this mechanism. In this paper, we will show how the two methods are consistent and mathematically similar.



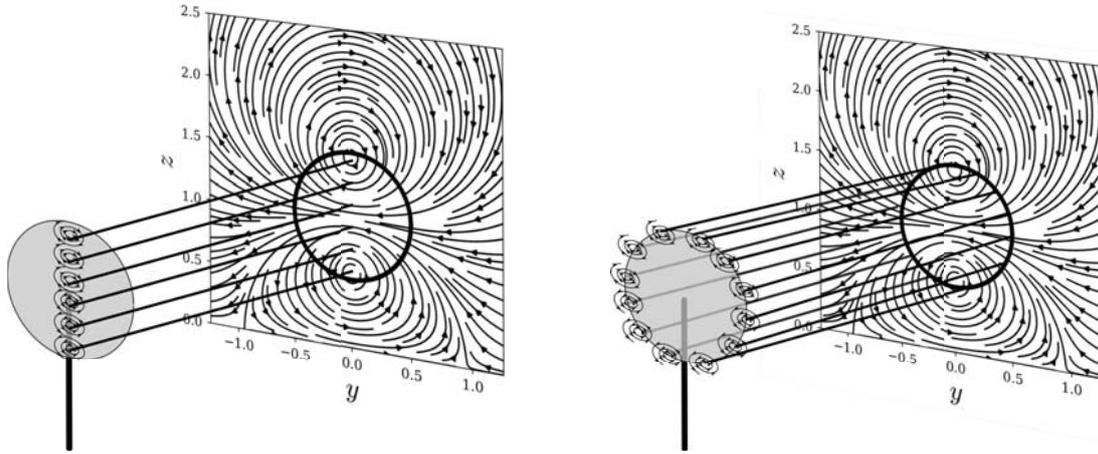


Figure 1. Schematic representing lines of streamwise vorticity from a yawed wind turbine with the induced velocities downstream. On the left, the original explanation of the mechanism is illustrated [1, 2, 3, 4]. The right figure shows the new mechanism proposed.

2. Formulations

2.1. Notations

Throughout the document we refer to x as the streamwise direction, y as the spanwise direction, and z as the wall-normal direction. The coordinate system is illustrated in Figure 2. We define the yaw angle with respect to the streamwise direction as θ_{yaw} . The wind turbine is yawed about the z -axis.

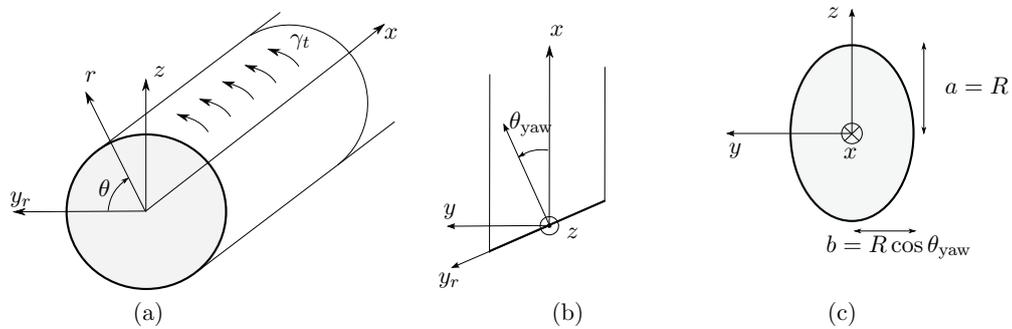


Figure 2. Coordinate system used in this document: (a) 3D perspective with the definition of θ in the rotor plane, (b) top view, and (c) view of the wake normal to the x direction.

In this work, we focus on the streamwise component (x) of the vorticity vector. This vorticity component is responsible for moving the wake in the spanwise directions. These motions cause the deformations that create the curled wake.

2.2. A line vortex in the streamwise direction (x)

The velocity vector induced at point \mathbf{x} by a point vortex at location \mathbf{x}_0 is given by:

$$\mathbf{u}(\mathbf{x}) = \frac{\Gamma}{2\pi} \frac{\mathbf{e}_x \times (\mathbf{x} - \mathbf{x}_0)}{\|\mathbf{x} - \mathbf{x}_0\|^2} \quad (1)$$

The vorticity distributions are regularized (or equivalently, the velocity fields) using the following Lamb-Oseen kernel:

$$K_\sigma(\mathbf{x}, \mathbf{x}_0, \sigma) = 1 - \exp(-\|\mathbf{x} - \mathbf{x}_0\|^2/\sigma^2) \quad (2)$$

where \mathbf{x} is a control point, \mathbf{x}_0 is the location of the vortex, and σ is a regularization parameter. Introducing the Lamb-Oseen regularization and the Cartesian coordinates leads to the following components of velocity:

$$u_y = \Gamma V_\sigma(y, z, y_0, z_0) = \frac{-(z - z_0)\Gamma}{2\pi((y - y_0)^2 + (z - z_0)^2)} \left(1 - e^{-((y - y_0)^2 + (z - z_0)^2)/\sigma^2}\right), \quad (3)$$

$$u_z = \Gamma W_\sigma(y, z, y_0, z_0) = \frac{(y - y_0)\Gamma}{2\pi((y - y_0)^2 + (z - z_0)^2)} \left(1 - e^{-((y - y_0)^2 + (z - z_0)^2)/\sigma^2}\right), \quad (4)$$

where V_σ and W_σ are regularized kernels of the Biot-Savart equation for the y and z direction, respectively.

2.3. Curled Wake Model: Original formulation

The original formulation in the curled wake model [4] assumes a vorticity distribution shed from a line across the rotor along the z direction (see Figure 1). The circulation distribution is defined as:

$$\frac{d\Gamma(z)}{dz} = -\Gamma_0 \frac{z}{R\sqrt{R^2 - z^2}}, \quad (5)$$

where R is the turbine radius, and z is the coordinate along the axis, about which the turbine is yawed. The spanwise velocities are the summation of all the vortices. This can be expressed as a continuous integral:

$$u_y(y, z) = -\int_{-R}^R V_\sigma(y, z, 0, z')\Gamma_0 \frac{z'}{R\sqrt{R^2 - z'^2}} dz' \quad (6)$$

$$u_z(y, z) = -\int_{-R}^R W_\sigma(y, z, 0, z')\Gamma_0 \frac{z'}{R\sqrt{R^2 - z'^2}} dz'. \quad (7)$$

These are the spanwise velocities responsible for creating the curled wake shape.

2.4. Semi-infinite vortex cylinder formulation

We now propose a new formulation based on a vortex cylinder. The semi-infinite vortex cylinder model introduced by Coleman is used [5]. The intensity of the vorticity sheet is tangential to the rotor and noted γ_t . A rigid wake assumption is used: the wake does not expand and keeps its cylindrical shape. This model has been later extended and applied to study wind turbines in yaw [9, 7, 8].

The vorticity in a plane normal to the free-stream direction, which from our assumption is also normal to the wake direction, forms an ellipse of parametric coordinates:

$$y_e = R \cos \theta \cos \theta_{\text{yaw}}, \quad z_e = R \sin \theta \quad (8)$$

where θ is a parameter going from 0 to 2π . The parameter, θ , corresponds to the azimuthal position in the rotor plane, and the coordinates, y_e and z_e , are the projection of the rotor circumference ($R \cos \theta, R \sin \theta$) in the plane under consideration.

At the rotor, vorticity is shed along the wake direction (mainly inducing swirl) and in the tangential direction (mainly inducing axial induction). We focus on the tangential vorticity, γ_t , which is the predominant component. The projection onto the x direction is:

$$\gamma_x = \gamma_t \sin \theta_{\text{yaw}} \sin \theta \quad (9)$$

The relationship between Γ_0 and γ_x is given by:

$$\gamma_t = \frac{\Gamma_0}{2 R \sin \theta_{\text{yaw}}}$$

The rotor disk circumference is located at:

$$x_r = -R \cos \theta \sin \theta_{\text{yaw}}, \quad y_r = R \cos \theta \cos \theta_{\text{yaw}}, \quad z_r = R \sin \theta. \quad (10)$$

Using this notation, the velocity field induced by the streamwise part of the vorticity, γ_x , at any point (i.e., x, y, z) of the domain is the superposition of all the vortices along the circumference:

$$\begin{aligned} u_y(x, y, z) &= \frac{R}{4\pi} \int_0^{2\pi} \gamma_x(\theta) \frac{(z_r - z)}{A_1(A_1 - A_2)} K_\sigma d\theta, \\ u_z(x, y, z) &= \frac{R}{4\pi} \int_0^{2\pi} \gamma_x(\theta) \frac{(y - y_r)}{A_1(A_1 - A_2)} K_\sigma d\theta \end{aligned} \quad (11)$$

with

$$A_1 = \sqrt{(x - x_r)^2 + (y - y_r)^2 + (z - z_r)^2}, \quad A_2 = x - x_r \quad (12)$$

2.5. Infinite vortex cylinder formulation

The infinite model corresponds to the far wake of the semi-infinite formulation. The two-dimensional representation of the vorticity presented in Equation 1 can directly be used by distributing the vortex points along the ellipse coordinates given in Equation 8:

$$u_y = R \int_0^{2\pi} \gamma_x(\theta) V_\sigma(y, z, R \cos \theta \cos \theta_{\text{yaw}}, R \sin \theta) d\theta \quad (13)$$

$$u_z = R \int_0^{2\pi} \gamma_x(\theta) W_\sigma(y, z, R \cos \theta \cos \theta_{\text{yaw}}, R \sin \theta) d\theta \quad (14)$$

We use the following transformation, $z' = R \sin \theta$ and $d\theta = dz' / (R \cos \theta)$, and split the integral into two parts.

$$u_y = \gamma_t \sin \theta_{\text{yaw}} \int_{-R}^R \left[V_\sigma \left(y, z, \cos \theta_{\text{yaw}} \sqrt{R^2 - z'^2}, z' \right) \right. \quad (15)$$

$$\left. + V_\sigma \left(y, z, -\cos \theta_{\text{yaw}} \sqrt{R^2 - z'^2}, z' \right) \right] \frac{z'}{\sqrt{R^2 - z'^2}} dz' \quad (16)$$

$$u_z = \gamma_t \sin \theta_{\text{yaw}} \int_{-R}^R \left[W_\sigma \left(y, z, \cos \theta_{\text{yaw}} \sqrt{R^2 - z'^2}, z' \right) \right. \quad (17)$$

$$\left. + W_\sigma \left(y, z, -\cos \theta_{\text{yaw}} \sqrt{R^2 - z'^2}, z' \right) \right] \frac{z'}{\sqrt{R^2 - z'^2}} dz' \quad (18)$$

These formulas show the velocity induced by the inclined vortex cylinder in a plane normal to the streamwise direction.

2.6. Link between the different formulations

The two formulations presented lead to a similar final form containing one integration. This integration should be performed numerically to obtain a final solution. Now, we show the equivalence between the equations that defines the original formulation of the curled wake model [4] and the new formulation. The original curled wake formulation yields the following spanwise velocity component (the z component is similar):

$$u_y = -\frac{\Gamma_0}{R} \int_{-R}^R V_\sigma(y, z, 0, z') \frac{z'}{\sqrt{R^2 - z'^2}} dz' \quad (19)$$

and the new formulation yields:

$$u_y = -\frac{\Gamma_0}{2R} \int_{-R}^R \left[V_\sigma \left(y, z, \cos \theta_{\text{yaw}} \sqrt{R^2 - z'^2}, z' \right) + V_\sigma \left(y, z, -\cos \theta_{\text{yaw}} \sqrt{R^2 - z'^2}, z' \right) \right] \frac{z'}{\sqrt{R^2 - z'^2}} dz' \quad (20)$$

Comparing the formulas, we can see that they are similar. There is an integration along the z direction with an exponential weighting. By making the approximation that $y_e = \cos \theta_{\text{yaw}} \sqrt{R^2 - z'^2} \approx 0$, Equation 20 collapses to be the same as Equation 19. It is remarkable to see that such different approaches yield a similar set of equations with some approximations. The reason why this approximation holds is because the strength of the circulation is strongest near the ellipse apex, ($|z| \approx R$). This approximation is useful to show the similarity between the two methods. However, the results show that these differences will produce different wake profiles, and the formulations are not equivalent without the approximation.

The link between the two-dimensional vorticity models can be established by lumping the vorticity onto the the z axis (this is the same as $y = 0$). At a given coordinate of the ellipse, z_e , the lumped vorticity at $\pm y_e$ is $\gamma_x R d\theta$. The vorticity has the same intensity on both sides of the z axis, and thus the lumped vorticity on the z axis is twice this value. Using $\theta = \arcsin z/R$ and $dz = R \cos \theta d\theta$ leads to the following lumped distribution of vorticity along the z -axis:

$$\frac{d\Gamma_x}{dz} = 2\gamma_t \sin \theta_{\text{yaw}} \frac{z}{\sqrt{R^2 - z^2}} = -\Gamma_0 \frac{z}{R \sqrt{R^2 - z^2}}, \quad (21)$$

where it is observed that an elliptical distribution similar to the one presented in the curled wake model, i.e., Equation 5, is obtained. The above expression may be interpreted as the projection of the ellipse vorticity onto the z -axis. This projection, and the sinusoidal variation of the axial vorticity, is such that most of the vorticity is concentrated at the two apex, $z = \pm R$, with a difference of sign on each side of the y -axis. The models are thus consistent and justify the simplified view of an elliptic distribution of vortices discussed in the literature [3].

3. Testing the new formulation in the curled wake model

We now test the new formulation for the spanwise velocities inside the curled wake model [4]. In Figure 3, results from the original curled wake model and the new curled wake formulations are compared to large-eddy simulations (LES) from Howland et al. [1] and the original implementation of the curled wake model [4]. The new model compares well with the original formulation and to the large-eddy simulations. Although general features of the flow are well captured by both models, there are some differences. The main difference is that the new formulation shows a more compact wake. It is not clear where this difference is coming from, but it seems that the wake expansion should be taken into account. Also, the new formulation is derived from an actuator disk. In the comparison, we use LES with an actuator line model. This could be a source of discrepancy between the two models.

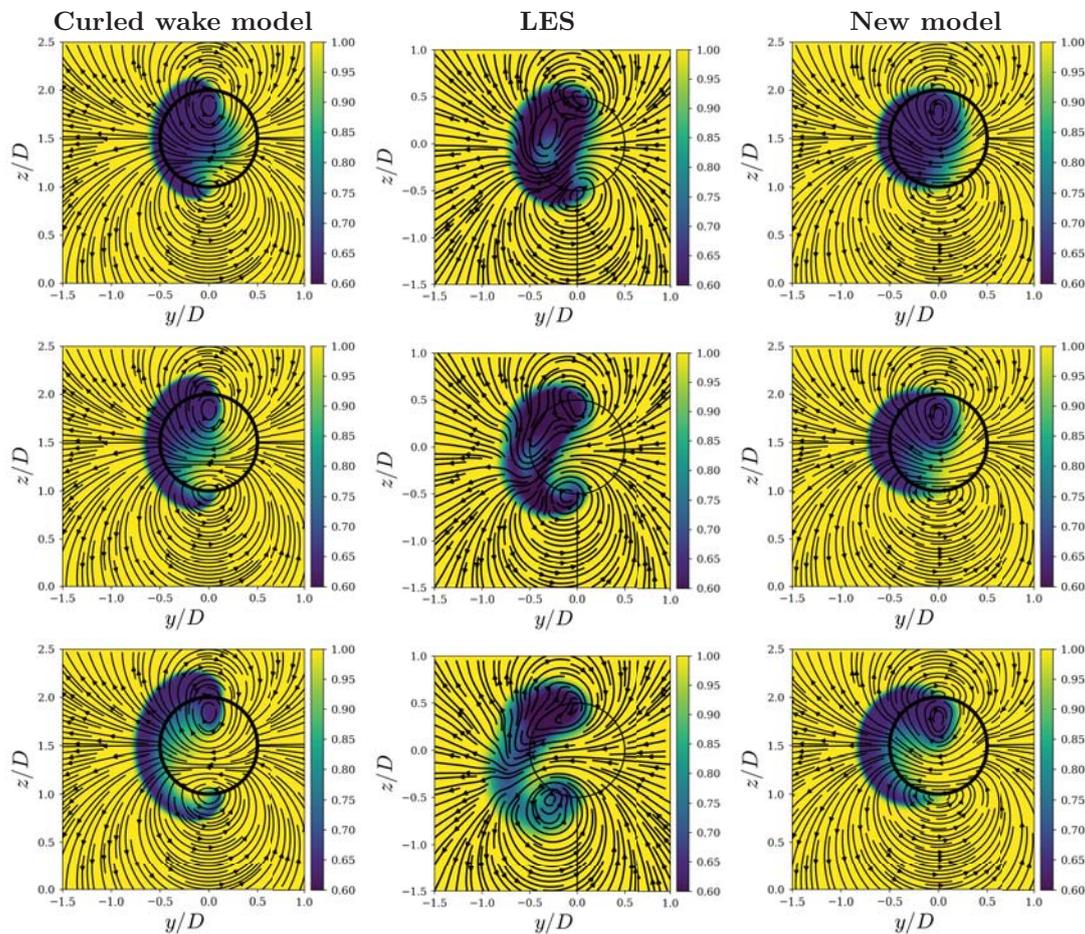


Figure 3. Comparison of streamwise velocity contours between large-eddy simulations of a wind turbine in 30° yaw, the original curled wake model, and the new model. The streamlines show the spanwise velocities. From top to bottom: downstream locations of 3D, 5D, and 8D. From left to right: the curled wake model, LES, and new model.

4. Discussions

The new curled wake model formulation presented in this work used the transverse velocity field from the vortex cylinder model. The new curled wake model uses the spanwise velocities from the vortex cylinder model in conjunction with the momentum transport equation to estimate the evolution of the streamwise velocity. The vortex cylinder model may be used to obtain the velocity in the streamwise direction, yet the results would be unrealistic. Indeed, the vortex cylinder model predicts a constant streamwise velocity within the vortex cylinder, a result contradicted by the LES results presented. The rigid wake assumption unrealistically freezes the geometry of the wake. The vorticity surface is not force free; in reality, the flow advects the vorticity surface, but this is prevented by the rigid-wake assumption. The curled wake model relaxes this assumption while solving for the moment transport in the streamwise direction. The curled wake model could further be improved by considering the motion of the vorticity surface as well. This will be considered in future work.

5. Conclusions

A new formulation of the curled wake model was introduced, inspired by a rigid vorticity wake model. The link between this new formulation and the original formulation was shown by considering the limit of the elliptical model. The two models provide a consistent formulation. The elliptic vorticity distribution used in the original formulation of the curled wake model is in agreement with the projection of the cylindrical vorticity distribution onto the yaw axis. The original formulation of the curled wake model seems to compare better to the results from LES. One possible explanation is that the vortices start out as a cylinder close to the rotor, and as they move downstream they form a shape that corresponds more to the vortices aligned with the yaw axis. Future work will consist of investigating the effect of having a finite number of blades on the model, and comparing the evolution of the vortices downstream.

Acknowledgments

The authors acknowledge helpful comments from Matt Churchfield and Paul Veers from NREL. This work was authored by the National Renewable Energy Laboratory, operated by Alliance for Sustainable Energy, LLC, for the U.S. Department of Energy (DOE) under Contract No. DE-AC36-08GO28308. Funding provided by the U.S. Department of Energy Office of Energy Efficiency and Renewable Energy Wind Energy Technologies Office. The views expressed in the article do not necessarily represent the views of the DOE or the U.S. Government. The U.S. Government retains and the publisher, by accepting the article for publication, acknowledges that the U.S. Government retains a nonexclusive, paid-up, irrevocable, worldwide license to publish or reproduce the published form of this work, or allow others to do so, for U.S. Government purposes.

References

- [1] Michael F. Howland, Juliaan Bossuyt, Luis A. Martínez-Tossas, Johan Meyers, and Charles Meneveau. Wake structure in actuator disk models of wind turbines in yaw under uniform inflow conditions. *Journal of Renewable and Sustainable Energy*, 8(4):043301, 2016.
- [2] Majid Bastankhah and Fernando Porté-Agel. Experimental and theoretical study of wind turbine wakes in yawed conditions. *Journal of Fluid Mechanics*, 806:506–541, 2016.
- [3] Carl R. Shapiro, Dennice F. Gayme, and Charles Meneveau. Modelling yawed wind turbine wakes: a lifting line approach. *Journal of Fluid Mechanics*, 841:R1, 2018.
- [4] L. A. Martínez-Tossas, J. Annoni, P. A. Fleming, and M. J. Churchfield. The aerodynamics of the curled wake: a simplified model in view of flow control. *Wind Energy Science*, 4(1):127–138, 2019.
- [5] R. P. Coleman, A. M. Feingold, and C. W. Stempin. Evaluation of the induced-velocity field of an idealized helicopter rotor. *NACA ARR No. L5E10*, pages 1–28, 1945.
- [6] W. Castles and H. L. Durham. Distribution of normal component of induced velocity in lateral plane of a lifting rotor. Technical report, NACA TN-3841, 1956.
- [7] E. Branlard and M. Gaunaa. Cylindrical vortex wake model: skewed cylinder, application to yawed or tilted rotors. *Wind Energy*, page (Online), 2015.
- [8] E. Branlard, M. Gaunaa, and E. Machefaux. Investigation of a new model accounting for rotors of finite tip-speed ratio in yaw or tilt. *Journal of Physics: Conference Series (Online)*, 524(1):1–11, 2014.
- [9] H. Glauert. A general theory of the autogyro. Technical report, NACA Reports and Memoranda No. 111, 1926.