Convex Relaxation of OPF in Multiphase Radial Networks with Wye and Delta Connections

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Optimal power flow (OPF)

Underlies many applications:

- Unit commitment, economic dispatch
- Market operations, market power analysis
- Placement and sizing of capacitors and storage
- Topology control, feeder reconfiguration
- Demand response
- Volt/var control

Optimal power flow (OPF)

Non-convex and hard to solve:

- Huge literature since Carpentier (1962)
- Common practice: **linearization**
  - DC power flow (Stott and Alsac 1974)
  - LinDistFlow (Baran and Wu 1989)
  - And more (Coffrin and Van Hentenryck 2014; Guggilam et al. 2015; Bolognani and Dorfler 2015).
- Last decade: **convex relaxation** (Jabr 2006; Bai et al. 2008; Lavaei and Low 2012; Molzahn et al. 2013; Farivar and Low 2013; Bose et al. 2015; Coffrin et al. 2016; Kocuk et al. 2016).

Convex relaxation of OPF in distribution networks

Radial, multiphase, wye + **delta**

$\underbrace{\text{Radial, multiphase, wye}}$

Dall'Anese et al. (2013) TSG
Gan and Low (2014) PSCC

$\underbrace{\text{Radial, multiphase, wye + delta}}$

This work

Outline

Preliminary: branch flow model, OPF, convex relaxation:

- Single phase
- Multiphase with wye connection only.

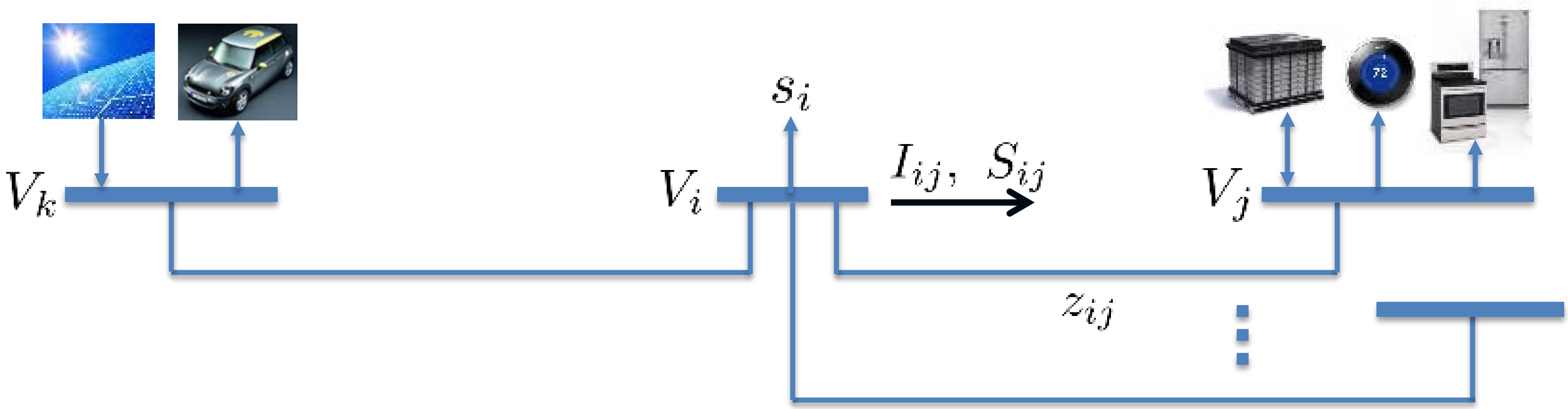
Modeling delta connection for OPF convex relaxation:

- Method 1: extended branch flow model (**EBFM**)
- Method 2: balanced voltage approximation (**BVA**).

Numerical results:

- Exactness; efficiency; optimality; accuracy of model.

Branch flow model (single phase)



- Auxiliary variables: $\begin{bmatrix} v_i & S_{ij} \\ S_{ij}^* & \ell_{ij} \end{bmatrix} = \begin{bmatrix} V_i \\ I_{ij} \end{bmatrix} \begin{bmatrix} V_i^* & I_{ij}^* \end{bmatrix}$

- Ohm's law: $V_i - V_j = z_{ij} I_{ij}$

  $\Rightarrow \quad v_j = v_i - (S_{ij} z_{ij}^* + z_{ij} S_{ij}^*) + z_{ij} \ell_{ij} z_{ij}^*$

- Power balance: $\sum_{k:k \to i} (S_{ki} - z_{ki} \ell_{ki}) = \sum_{j:i \to j} S_{ij} + s_i$

OPF (single phase)

x

$$\min \quad f(s)$$

$$\text{over} \quad (s, v, \ell, S)$$

$$\text{s.t.} \quad \underline{v}_i \le v_i \le \overline{v}_i, \quad s_i \in \mathcal{S}_i, \quad \forall i$$

Branch flow model

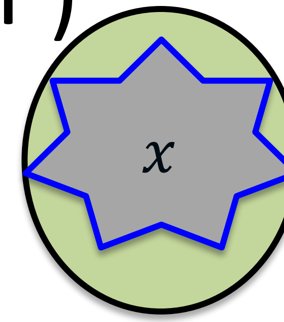
$$v_j = v_i - (S_{ij} z_{ij}^* + z_{ij} S_{ij}^*) + z_{ij} \ell_{ij} z_{ij}^*, \ \forall i \to j$$

$$\sum_{k:k\to i} (S_{ki} - z_{ki}\ell_{ki}) = \sum_{j:i\to j} S_{ij} + s_i, \quad \forall i$$

$$\begin{bmatrix} v_i & S_{ij} \\ S_{ij}^* & \ell_{ij} \end{bmatrix} \succeq 0, \quad \text{rank}\left(\begin{bmatrix} v_i & S_{ij} \\ S_{ij}^* & \ell_{ij} \end{bmatrix}\right) = 1, \quad \forall i \to j$$

Non-convex

Convex relaxation as a second-order cone program (SOCP)



$$\min \quad f(s)$$

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$$\text{s.t.} \quad \underline{v}_i \le v_i \le \overline{v}_i, \quad s_i \in \mathcal{S}_i, \quad \forall i$$

Branch flow model

$$v_j = v_i - (S_{ij} z_{ij}^* + z_{ij} S_{ij}^*) + z_{ij} \ell_{ij} z_{ij}^*, \ \forall i \to j$$

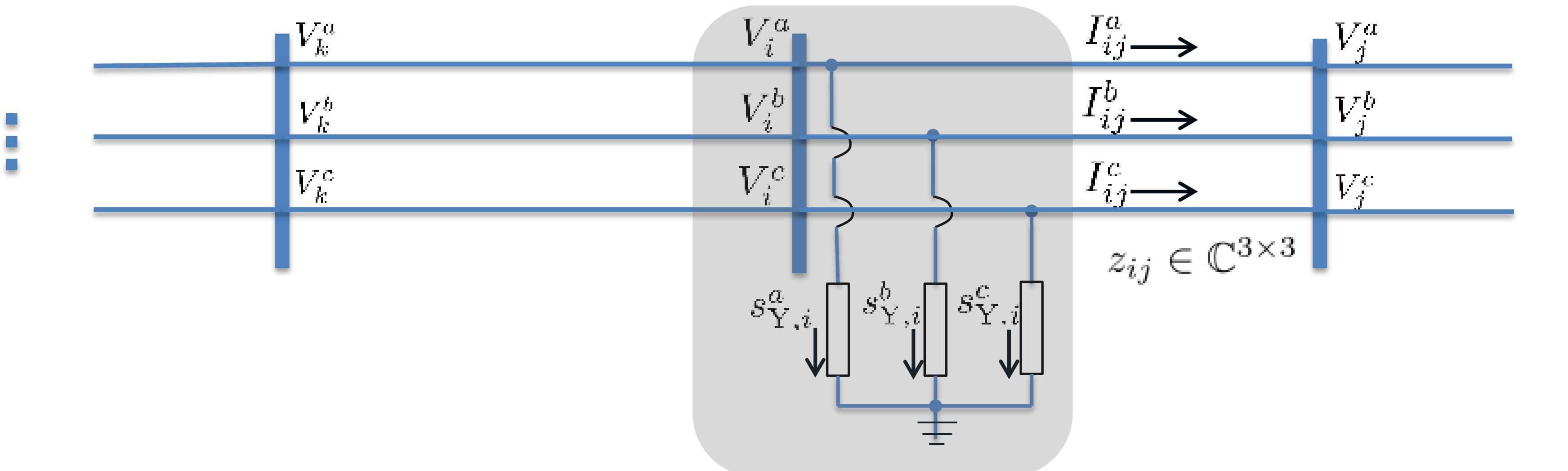
$$\sum_{k:k\to i} (S_{ki} - z_{ki}\ell_{ki}) = \sum_{j:i\to j} S_{ij} + s_i, \quad \forall i$$

$$\begin{bmatrix} v_i & S_{ij} \\ S_{ij}^* & \ell_{ij} \end{bmatrix} \succeq 0, \quad \cancel{\text{rank}\left(\begin{bmatrix} v_i & S_{ij} \\ S_{ij}^* & \ell_{ij} \end{bmatrix}\right)} = 1, \quad \forall i \to j$$

Equivalent to a second-order cone

Farivar et al. (2011) SGComm
Farivar and Low (2013) TPWRS

Branch flow model (multiphase, wye)



- Auxiliary variables: $\begin{bmatrix} v_i & S_{ij} \\ S_{ij}^H & \ell_{ij} \end{bmatrix} = \begin{bmatrix} V_i \\ I_{ij} \end{bmatrix} \begin{bmatrix} V_i^H & I_{ij}^H \end{bmatrix}$

- Ohm's law: $V_i - V_j = z_{ij} I_{ij}$

  $\Rightarrow \quad v_j = v_i - \left(S_{ij} z_{ij}^H + z_{ij} S_{ij}^H\right) + z_{ij} \ell_{ij} z_{ij}^H$

- Power balance: $\sum_{k:k\to i} \operatorname{diag}\left(S_{ki} - z_{ki}\ell_{ki}\right) = \sum_{j:i\to j} \operatorname{diag}\left(S_{ij}\right) + s_{Y,i}$

OPF (multiphase, wye)

$$\min \quad f(s_Y)$$

$$\text{over} \quad (s_Y, v, \ell, S)$$

$$\text{s.t.} \quad \underline{v}_i \leq v_i \leq \overline{v}_i, \quad s_{Y,i} \in \mathcal{S}_{Y,i}, \quad \forall i$$

Branch flow model

$$v_j = v_i - (S_{ij} z_{ij}^H + z_{ij} S_{ij}^H) + z_{ij} \ell_{ij} z_{ij}^H, \ \forall i \to j$$

$$\sum_{k:k\to i} \operatorname{diag}(S_{ki} - z_{ki}\ell_{ki}) = \sum_{j:i\to j} \operatorname{diag}(S_{ij}) + s_{Y,i}, \quad \forall i$$

$$\begin{bmatrix} v_i & S_{ij} \\ S_{ij}^H & \ell_{ij} \end{bmatrix} \succeq 0, \quad \operatorname{rank}\left(\begin{bmatrix} v_i & S_{ij} \\ S_{ij}^H & \ell_{ij} \end{bmatrix}\right) = 1, \quad \forall i \to j$$

Non-convex

Convex relaxation as a semidefinite program (SDP)

$$\min \quad f\left(s_{\mathrm{Y}}\right)$$

$$\text{over} \quad \left(s_{\mathrm{Y}}, v, \ell, S\right)$$

$$\text{s.t.} \quad \underline{v}_i \le v_i \le \overline{v}_i, \quad s_{\mathrm{Y},i} \in \mathcal{S}_{\mathrm{Y},i}, \quad \forall i$$

Branch flow model

$$v_j = v_i - \left(S_{ij} z_{ij}^H + z_{ij} S_{ij}^H\right) + z_{ij} \ell_{ij} z_{ij}^H, \ \forall i \to j$$

$$\sum_{k:k\to i} \operatorname{diag}\left(S_{ki} - z_{ki}\ell_{ki}\right) = \sum_{j:i\to j} \operatorname{diag}\left(S_{ij}\right) + s_{\mathrm{Y},i}, \quad \forall i$$

$$\begin{bmatrix} v_i & S_{ij} \\ S_{ij}^H & \ell_{ij} \end{bmatrix} \succeq 0, \quad \cancel{\operatorname{rank}\left(\begin{bmatrix} v_i & S_{ij} \\ S_{ij}^H & \ell_{ij} \end{bmatrix}\right) = 1}, \quad \forall i \to j$$

6x6 semidefinite matrix

Gan and Low (2014) PSCC

Outline

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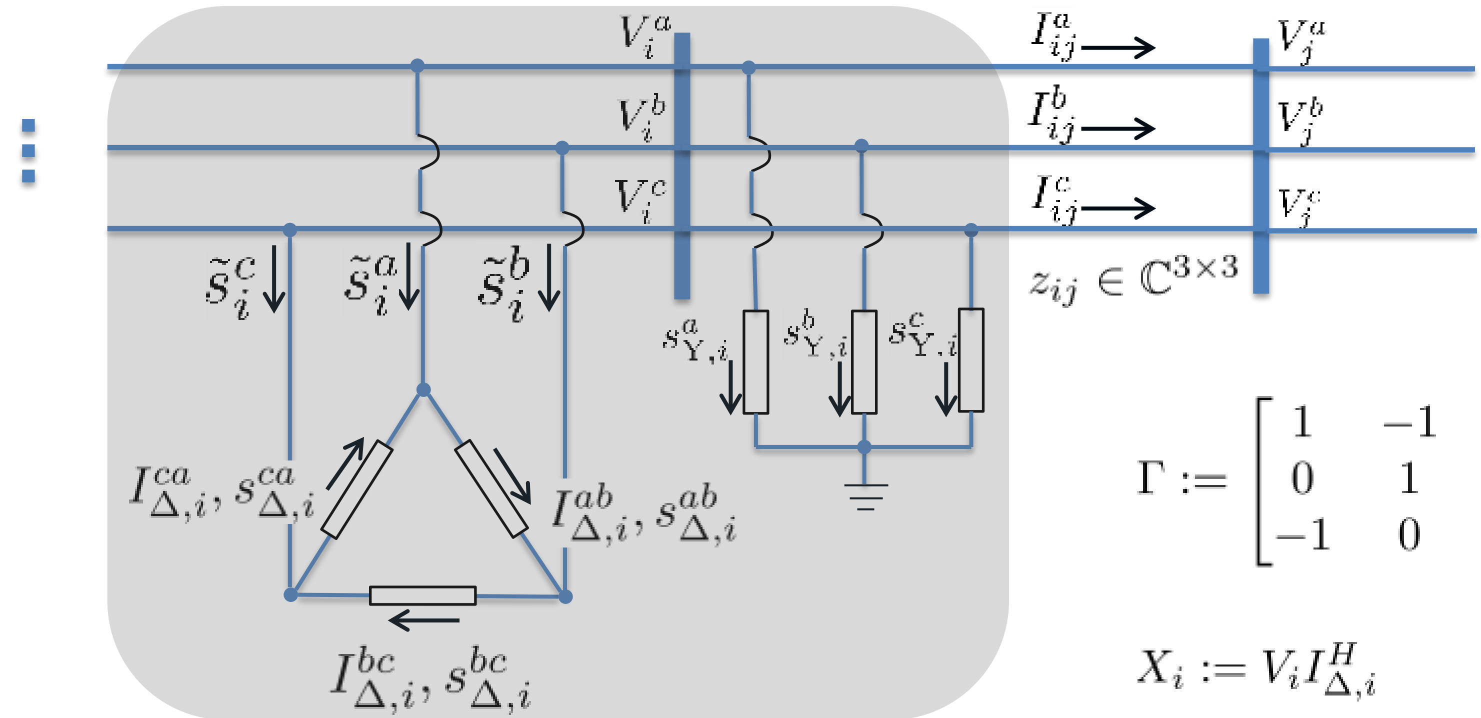
Modeling delta connection for SDP relaxation:

- Method 1: extended branch flow model (**EBFM**)
- Method 2: balanced voltage approximation (**BVA**).

Numerical results:

- Exactness; efficiency; optimality; accuracy of model.

Extended branch flow model (**EBFM**)



$$\Gamma := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$X_i := V_i I_{\Delta,i}^H$$

$$s_{\Delta,i} = \begin{bmatrix} (V_i^a - V_i^b)(I_{\Delta,i}^{ab})^* \\ (V_i^b - V_i^c)(I_{\Delta,i}^{bc})^* \\ (V_i^c - V_i^a)(I_{\Delta,i}^{ca})^* \end{bmatrix} = \mathrm{diag}\left(\Gamma V_i I_{\Delta,i}^H\right) = \mathrm{diag}(\Gamma X_i)$$

$$\mathrm{diag}(X_i \Gamma) = \mathrm{diag}\left(V_i I_{\Delta,i}^H \Gamma\right) = \begin{bmatrix} V_i^a (I_{\Delta,i}^{ab} - I_{\Delta,i}^{ca})^* \\ V_i^b (I_{\Delta,i}^{bc} - I_{\Delta,i}^{ab})^* \\ V_i^c (I_{\Delta,i}^{ca} - I_{\Delta,i}^{bc})^* \end{bmatrix} = \tilde{s}_i$$

Implicit Δ-Y **power** transform

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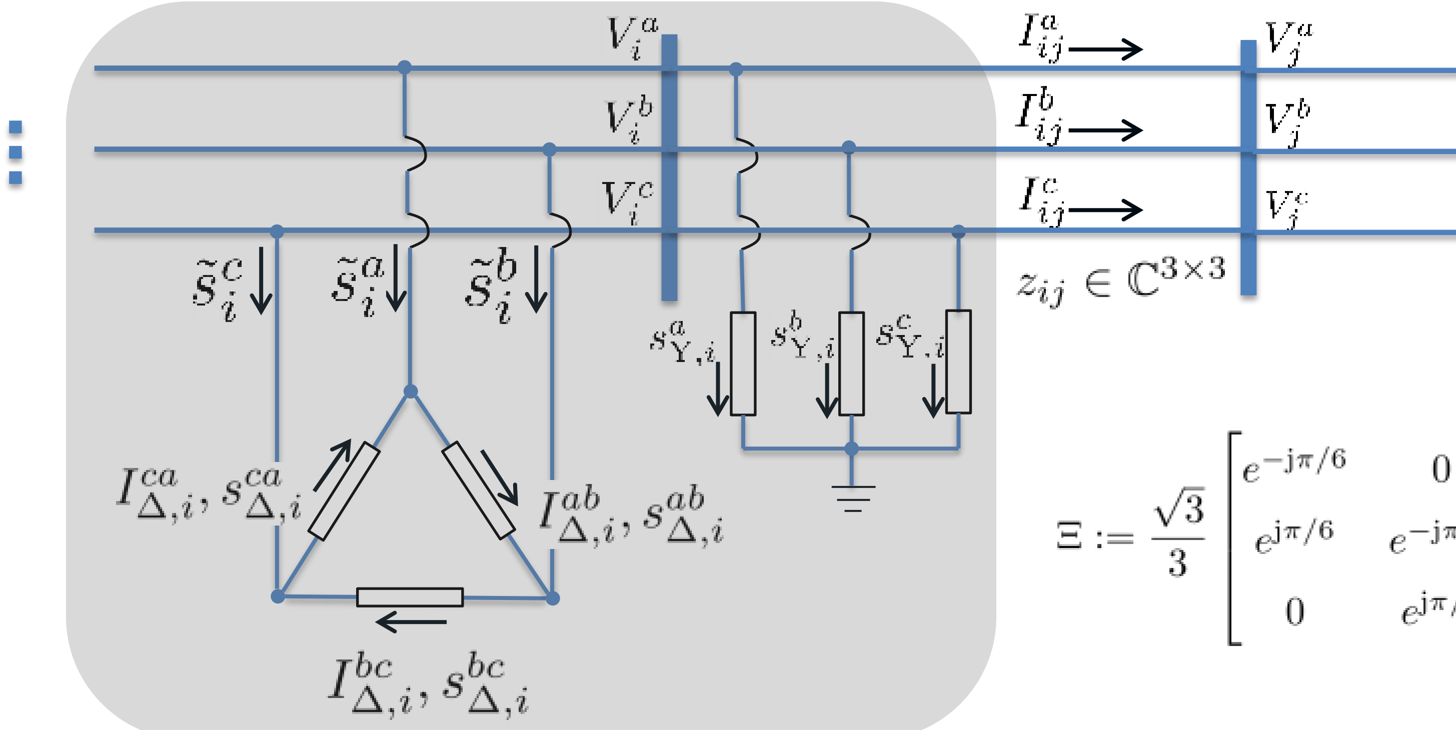
OPF (multiphase, wye+delta, **EBFM**)

$$
\begin{aligned}
\min \quad & f\left(s_{\mathrm{Y}}, s_{\Delta}\right) \\
\text{over} \quad & \left(s_{\mathrm{Y}}, s_{\Delta}, v, \ell, S, X, \rho\right) \\
\text{s.t.} \quad & \underline{v}_i \le v_i \le \overline{v}_i, \quad s_{\mathrm{Y},i} \in \mathcal{S}_{\mathrm{Y},i}, \quad s_{\Delta,i} \in \mathcal{S}_{\Delta,i}, \quad \forall i \\
& v_j = v_i - \left(S_{ij} z_{ij}^H + z_{ij} S_{ij}^H\right) + z_{ij} \ell_{ij} z_{ij}^H, \ \forall i \to j \\
& \sum_{k:k\to i} \operatorname{diag}\left(S_{ki} - z_{ki}\ell_{ki}\right) = \sum_{j:i\to j} \operatorname{diag}\left(S_{ij}\right) + s_{\mathrm{Y},i} + \operatorname{diag}(X_i \Gamma), \quad \forall i \\
& s_{\Delta,i} = \operatorname{diag}(\Gamma X_i), \quad \forall i \\
& \begin{bmatrix} v_i & S_{ij} \\ S_{ij}^H & \ell_{ij} \end{bmatrix} \succeq 0, \quad \cancel{\operatorname{rank}\left(\begin{bmatrix} v_i & S_{ij} \\ S_{ij}^H & \ell_{ij} \end{bmatrix}\right) = 1}, \quad \forall i \to j \\
& \begin{bmatrix} v_i & X_i \\ X_i^H & \rho_i \end{bmatrix} \succeq 0, \quad \cancel{\operatorname{rank}\left(\begin{bmatrix} v_i & X_i \\ X_i^H & \rho_i \end{bmatrix}\right) = 1}, \quad \forall i
\end{aligned}
$$

SDP relaxation

Non-convex

Balanced voltage approximation (**BVA**)



$$\Xi := \frac{\sqrt{3}}{3}\begin{bmatrix} e^{-\mathrm{j}\pi/6} & 0 & e^{\mathrm{j}\pi/6} \\ e^{\mathrm{j}\pi/6} & e^{-\mathrm{j}\pi/6} & 0 \\ 0 & e^{\mathrm{j}\pi/6} & e^{-\mathrm{j}\pi/6} \end{bmatrix}$$

Approximation: $$\frac{V_i^a}{V_i^b} = \frac{V_i^b}{V_i^c} = \frac{V_i^c}{V_i^a} = e^{\mathrm{j}2\pi/3}, \quad \forall i$$

$$\Rightarrow \quad \tilde{s}_i = \Xi s_{\Delta,i}$$

Explicit (approximate)
Δ-Y **power** transform

OPF (multiphase, wye+delta, **BVA**)

$$\min \quad f\left(s_{\mathrm{Y}}, s_{\Delta}\right)$$

$$\text{over} \quad \left(s_{\mathrm{Y}}, s_{\Delta}, v, \ell, S\right)$$

$$\text{s.t.} \quad \underline{v}_i \leq v_i \leq \overline{v}_i, \quad s_{\mathrm{Y},i} \in \mathcal{S}_{\mathrm{Y},i}, \quad s_{\Delta,i} \in \mathcal{S}_{\Delta,i}, \quad \forall i$$

$$v_j = v_i - \left(S_{ij} z_{ij}^H + z_{ij} S_{ij}^H\right) + z_{ij} \ell_{ij} z_{ij}^H, \ \forall i \rightarrow j$$

$$\sum_{k:k\rightarrow i} \operatorname{diag}\left(S_{ki} - z_{ki}\ell_{ki}\right) = \sum_{j:i\rightarrow j} \operatorname{diag}\left(S_{ij}\right) + s_{\mathrm{Y},i} + \Xi s_{\Delta,i}, \quad \forall i$$

SDP relaxation

$$\begin{bmatrix} v_i & S_{ij} \\ S_{ij}^H & \ell_{ij} \end{bmatrix} \succeq 0, \quad \cancel{\operatorname{rank}\left(\begin{bmatrix} v_i & S_{ij} \\ S_{ij}^H & \ell_{ij} \end{bmatrix}\right) = 1}, \quad \forall i \rightarrow j$$

Non-convex

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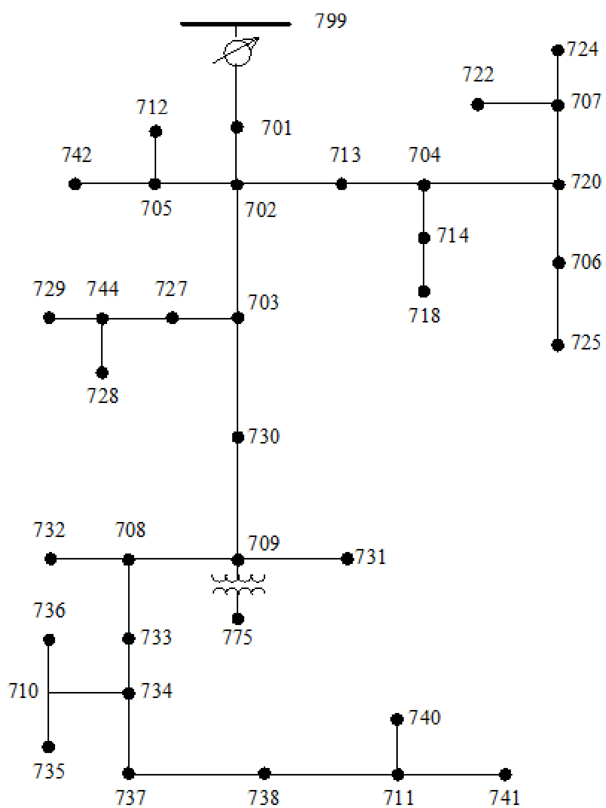
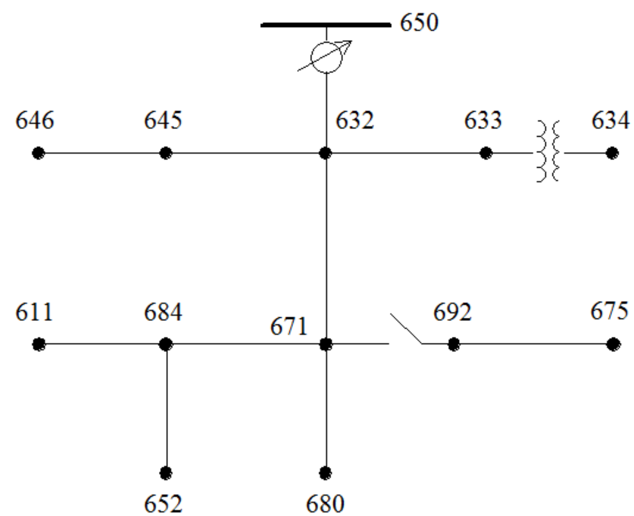
Setup for numerical studies



IEEE 37-bus feeder
All loads are delta-connected.



IEEE 13-bus feeder
wye : delta = 55 : 45

OPF objective: substation power tracking + utility of demand response

SDP solver: SeDuMi on CVX

Exactness/Feasibility

small ratio: nearly rank-1

large ratio (EBFM): violate rank-1

network	voltage	method	(v, S, ℓ)-ratio	(v, X, ρ)-ratio
IEEE 13	2%	EBFM	1.028×10^{-7}	0.9893
		BVA	2.443×10^{-7}	-
	5%	EBFM	1.194×10^{-7}	0.9577
		BVA	1.733×10^{-7}	-
IEEE 37	2%	EBFM	1.111×10^{-6}	0.9352
		BVA	8.605×10^{-3}	-
	5%	EBFM	1.155×10^{-6}	0.9094
		BVA	9.494×10^{-8}	-

$|\lambda_2/\lambda_1|$ $\uparrow$ $\qquad$ $|\lambda_2/\lambda_1|$ $\uparrow$

$$\operatorname{rank}\left(\begin{bmatrix} v_i & S_{ij} \\ S_{ij}^H & \ell_{ij} \end{bmatrix}\right) = 1, \quad \operatorname{rank}\left(\begin{bmatrix} v_i & X_i \\ X_i^H & \rho_i \end{bmatrix}\right) = 1$$

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Efficiency and optimality

network	voltage	method	time (s)	Obj_{opt}	$\text{Obj}_{\text{no control}}$
IEEE 13	2%	EBFM	2.246	10.65	106.6
		BVA	1.763	10.93	
	5%	EBFM	2.200	10.55	105.0
		BVA	2.075	10.83	
IEEE 37	2%	EBFM	9.719	6.348	64.59
		BVA	5.366	7.019	
	5%	EBFM	9.438	6.271	63.42
		BVA	3.136	6.379	

- SDP-BVA can be solved faster than SDP-EBFM.
- Both EBFM and BVA achieve significant cost reduction over "no control."
- SDP-EBFM provides lower bound of objective; SDP-BVA achieves small optimality gap.

Accuracy of model

Voltage recovered from SDP vs. voltage solved by OpenDSS:

network	voltage	method	RMSE (pu)	MAX (pu)
IEEE 13	2%	EBFM	6.953×10^{-3}	1.467×10^{-2}
		BVA	1.488×10^{-4}	2.915×10^{-4}
	5%	EBFM	6.753×10^{-3}	1.424×10^{-2}
		BVA	1.357×10^{-4}	2.673×10^{-4}
IEEE 37	2%	EBFM	6.528×10^{-3}	1.113×10^{-2}
		BVA	2.547×10^{-4}	9.833×10^{-4}
	5%	EBFM	6.343×10^{-3}	1.081×10^{-2}
		BVA	2.855×10^{-4}	5.496×10^{-4}

- BVA is accurate.
- EBFM is less accurate due to violation of rank-1 constraint.

Summary

Modeling delta connection in multiphase OPF and its SDP relaxation:

- Method 1: Extended branch flow model (EBFM)
- Method 2: Balanced voltage approximation (BVA).

Numerical results: BVA strikes a good balance among feasibility, accuracy, efficiency, and optimality.